JOURNAL OF APPROXIMATION THEORY 37, 246–250 (1983)

Remarks on Alice Roth's Fusion Lemma

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Communicated by Richard S. Varga

Received February 3, 1982

1. INTRODUCTION

In the study of rational approximation in the complex plane a fundamental role is played by the so-called "*Fusion Lemma*" of Alice Roth (see [1, p. 113 ff]). Let K_1, K_2, k be compact sets in $\mathbb{C}$ with $K_1 \cap K_2 = \varnothing$. Then there exists a constant A depending on K_1, K_2 only such that: Given two rational functions r_1, r_2 with

$$|r_1(z) - r_2(z)| \leqslant 1 \qquad (z \in k),$$

there exists another rational function r with

$$|r(z) - r_1(z)| \leqslant A \quad (z \in K_1 \cup k) \qquad \text{and} \qquad |r(z) - r_2(z)| \leqslant A \quad (z \in K_2 \cup k). \tag{1.1}$$

It is easily seen that the important case is when k meets both K_1 and K_2. We can say, then, that r connects r_1 on K_1 to r_2 on K_2 over the "bridge" k.

It is not as clear, however, whether the following stronger form of the Fusion Lemma is true.

PROPOSITION P. *Let K_1, K_2 be compact sets in $\mathbb{C}$ with $k = K_1 \cap K_2 \neq \varnothing$. Assume that two rational functions r_1, r_2 are given with $|r_1(z) - r_2(z)| \leqslant 1$ for $z \in k$. Then there is a rational function r with*

$$|r(z) - r_1(z)| \leqslant A \quad (z \in K_1) \qquad \textit{and} \qquad |r(z) - r_2(z)| \leqslant A \quad (z \in K_2), \tag{1.2}$$

where A depends on K_1, K_2 only.

It is the purpose of this note to study this proposition. It will turn out that it is false even in very simple geometric situations. In Section 3 we remark on the simultaneous approximation of two holomorphic functions in two adjacent regions.

0021-9045/83 $3.00

2. Study of Proposition P

First, we notice that *Proposition* P *is true if*

$$(K_1 \backslash k)^- \text{ and } (K_2 \backslash k)^- \text{ are disjoint sets.} \tag{2.1}$$

(Here M^- means the closure of M.) Indeed, if we use the sets $(K_1 \backslash k)^-$ and $(K_2 \backslash k)^-$ for K_1 and K_2 in Roth's Lemma, we get (1.2) from (1.1). Criterion (2.1) can be applied, for example, if K_1 and K_2 are two overlapping rectangles such that $K_1 \backslash (K_1 \cap K_2)$ and $K_2 \backslash (K_1 \cap K_2)$ are at positive distance.

Second, *Proposition* P *is also true if*

$$\begin{aligned} &K_1 \cap K_2 \text{ is a finite point set } \{z_1, z_2, \ldots, z_k\}, \\ &K_1 \cup K_2 \text{ has a complement } (K_1 \cup K_2)^c \text{ consisting} \\ &\quad \text{of a finite number of components.} \end{aligned} \tag{2.2}$$

To prove this, assume first that r_1, r_2 have no poles on K_1, K_2, respectively. Find a polynomial p with $p(z_j) = r_2(z_j) - r_1(z_j)$ $(j = 1, 2, \ldots, k)$ satisfying $|p(z)| \leqslant M$ $(z \in K_1 \cup K_2)$ for a suitable constant M depending on K_1, K_2 only. The function

$$\begin{aligned} F(z) &= r_1(z) && (z \in K_1), \\ &= r_2(z) - p(z) && (z \in K_2), \end{aligned}$$

will then be holomorphic on $(K_1 \cup K_2)^0$ and continuous on $K_1 \cup K_2$. Using the second part of (2.2), we find a rational function r with $|r(z) - F(z)| \leqslant 1$ $(z \in K_1 \cup K_2)$ from which (1.2) follows with $A = M + 1$.

If r_1, r_2 have poles on K_1, K_2, let h be the sum of the singular parts of r_1 on K_1 plus the sum of the singular parts of r_2 on $K_2 \backslash k$. Then $r_1 - h$ will be rational and holomorphic on K_1 and $r_2 - h$ on K_2. Notice that r_1 and r_2 have the same singular parts on k since $\|r_1 - r_2\|_k \leqslant 1$. Now, since Proposition P is already proved for $r_1 - h$ and $r_2 - h$ instead of r_1, r_2, we get a rational function r such that

$$|(r+h) - r_1| \leqslant A \quad \text{on} \quad K_1 \qquad \text{and} \qquad |(r+h) - r_2| \leqslant A \quad \text{on} \quad K_2$$

as was claimed above.

That Proposition P is not true in general was pointed out first by Paul Gauthier (see [1, p. 116]). The two compact sets K_1, K_2 were, however, rather complicated, and the question arose whether it *was* true for a simple geometric configuration. It is our main object to show that Proposition P is false even if K_1 and K_2 are two adjoining squares.

THEOREM 1. *Let K_1 and K_2 be the two closed squares in the upper and lower half planes, respectively, which have $I=[-1,+1]$ as a common edge. Then Proposition* P *is false.*

For the proof we shall need the following:

LEMMA. *Given $M>0$ there exists a polynomial P with*

$$P(0)=0, \qquad |P(x)|\leqslant 1 \qquad \textit{for} \quad x\in[-1,+1]$$

and

$$\left|\int_{-1}^{1}\frac{P(x)}{x}\,dx\right|\geqslant M.$$

Proof. Let $0<\delta<1$ and consider the auxiliary function

$$\begin{aligned} h_\delta(x) &= 1/|x| \qquad \text{for} \quad \delta\leqslant|x|\leqslant 1,\\ &= 1/\delta \qquad \text{for} \quad 0\leqslant|x|\leqslant\delta. \end{aligned}$$

Choose δ so small that $\int_{-1}^{1} h_\delta(x)\,dx\geqslant 2M+2$, and keep δ fixed. By Weierstrass' theorem, there is a polynomial p with $|h_\delta(x)-p(x)|\leqslant 1$ for $|x|\leqslant 1$. We then have

$$|p(x)|\leqslant 1/|x|+1\leqslant 2/|x| \qquad \text{for} \quad |x|\leqslant 1$$

and

$$\left|\int_{-1}^{1} p(x)\,dx\right|\geqslant\int_{-1}^{1} h_\delta(x)\,dx-2\geqslant 2M.$$

If therefore $P(x)=xp(x)/2$, we get $P(0)=0$, $|P(x)|\leqslant 1$ for $|x|\leqslant 1$ and $|\int_{-1}^{1}(P(x)/x)\,dx|=\frac{1}{2}|\int_{-1}^{1}p(x)\,dx|\geqslant M$.

Proof of Theorem 1. Assume that Proposition P holds for the compact squares K_1 and K_2 with an absolute constant A. Choose a polynomial P according to our lemma for an $M>24A$, and consider the functions P on K_1, 0 on K_2, so that $|P(x)|\leqslant 1$ for $x\in k=K_1\cap K_2=I$. Assume now that there exists a function H holomorphic on $K_1\cup K_2$ such that

$$|H(z)-P(z)|\leqslant A \quad (z\in K_1) \qquad \text{and} \qquad |H(z)-0|\leqslant A \quad (z\in K_2)$$

as required by Proposition P. Putting $H_1=H-H(0)$ we have $H_1(0)=0$ and

$$|H_1(z)-P(z)|\leqslant 2A \quad (z\in K_1) \qquad \text{and} \qquad |H_1(z)|\leqslant 2A \quad (z\in K_2).$$

Now put $C_1 = \partial K_1 \cap \{z: \operatorname{Im} z > 0\}$ and $C_2 = \partial K_2 \cap \{z: \operatorname{Im} z < 0\}$. By Cauchy's theorem

$$0 = \int_{C_1 \cup C_2} \frac{H_1(z)}{z}\, dz = \int_{C_1} \frac{H_1(z) - P(z)}{z}\, dz + \int_{C_1} \frac{P(z)}{z}\, dz + \int_{C_2} \frac{H_1(z)}{z}\, dz.$$

The first and the last integrals on the right are in absolute value $\leqslant 2A \cdot 6$ each, whereas,

$$\int_{C_1} \frac{P(z)}{z}\, dz = \int_{C_1 \cup I} \frac{P(z)}{z}\, dz - \int_I \frac{P(z)}{z}\, dz = 0 - \int_I \frac{P(z)}{z}\, dz.$$

Hence we would get $|\int_I (P(z)/z)\, dz| \leqslant 24A$ against the choice of the polynomial P.

3. Approximation of Holomorphic Functions in Adjacent Regions

We shall finally show that Proposition P can be partially saved if in (1.2) we stay away from the common part $k = K_1 \cap K_2$.

THEOREM 2. *We assume for simplicity that K_1 and K_2 are two compact sets in $\mathbb{C}$ bounded by two rectifiable Jordan curves, and that $K_1 \cap K_2$ is a rectifiable Jordan arc γ of length L. Let f_j be continuous on K_j and holomorphic on K_j^0 and assume that $|f_1(z) - f_2(z)| \leqslant \varepsilon$ for $z \in \gamma$. Then there is a function F holomorphic on $(K_1 \cup K_2)^0$ such that*

$$|F(z) - f_j(z)| \leqslant L\varepsilon/2\pi d_z \qquad \text{for} \quad z \in K_j^0, \quad j = 1, 2, \tag{3.1}$$

where d_z denotes the distance from z to γ.

For $\varepsilon = 0$ we obtain the principle of continuity. Equation (3.1) expresses the fact that F will approximate f_j in K_j^0 if z stays away from $K_1 \cap K_2$. Section 2 showed that the factor of ε cannot be chosen independently of z.

Proof. We put $C_1 = \partial K_1 \backslash \gamma$ and $C_2 = \partial K_2 \backslash \gamma$ so that $C_1 \cup C_2 = C$ is the positively oriented boundary of $K_1 \cup K_2$. Let

$$F(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f_1(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{C_2} \frac{f_2(\zeta)}{\zeta - z}\, d\zeta \qquad (z \notin C)$$

which is holomorphic in $(K_1 \cup K_2)^0$. If $z \in K_1^0$,

$$F(z) = \frac{1}{2\pi i} \int_{C_1 \cup \gamma} \frac{f_1(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{C_2 \cup -\gamma} \frac{f_2(\zeta)}{\zeta - z} \, d\zeta$$

$$- \frac{1}{2\pi i} \int_\gamma \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} \, d\zeta$$

$$= f_1(z) + 0 - \frac{1}{2\pi i} \int_\gamma,$$

and the latter term is in absolute value $\leqslant (1/2\pi)(\varepsilon/d_z)L$; similarly, if $z \in K_2^0$.

Reference

1. D. Gaier, "Vorlesungen über Approximation im Komplexen," Birkhäuser, Basel/Boston/Stuttgart, 1980.